

Liouville theorems for stationary flows of shear thickening fluids in $2D$ *

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Abstract

In this paper we consider the entire weak solutions u of the equations for stationary flows of shear thickening fluids in the plane and prove Liouville theorems under the conditions on the finiteness of energy and under the integrability condition of the solutions.

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1 Introduction

In this paper, we prove different types of Liouville theorems for the entire weak solutions $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ of the following system

$$(1.1) \quad \begin{cases} -\operatorname{div}[T(\varepsilon(u))] + u^k \partial_k u + D\pi = 0, \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^2, \end{cases}$$

which describes the stationary flow of an incompressible generalized Newtonian fluid. In equation (1.1), u denotes the velocity field, π the pressure function, $u^k \partial_k u$ the convective term, and T represents the stress tensor. As usual $\varepsilon(u)$ is the symmetric derivative of u , i.e. $\varepsilon(u) = \frac{1}{2}(Du + (Du)^T) = \frac{1}{2}(\partial_i u^k + \partial_k u^i)_{1 \leq i, k \leq 2}$.

We assume that the stress tensor is the gradient of a potential $H : S^2 \rightarrow \mathbb{R}$ defined on the space S^2 of all symmetric (2×2) matrices of the following form

$$(1.2) \quad H(\varepsilon) = h(|\varepsilon|),$$

where h is a nonnegative function of class C^2 . Thus

$$(1.3) \quad T(\varepsilon) = DH(\varepsilon) = \mu(|\varepsilon|)\varepsilon, \quad \mu(t) = \frac{h'(t)}{t}.$$

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Here μ denotes the viscosity coefficient. In case of generalized Newtonian fluids, it may depend on $\varepsilon(u)$. This means that it depends on the motion of the fluids. If $\mu(t)$ is an increasing function, the fluid is called shear thickening one. If $\mu(t)$ is a decreasing function, the fluid is shear thinning. If $\mu(t)$ is a constant, then the fluid is Newtonian and (1.1) reduces to the stationary Navier-Stokes equations for incompressible Newtonian fluids. For the further mathematical and physical explanations, we refer to Ladyzhenskaya [Lad69], Galdi [Gal94a, Gal94b], Malek, Necas, Rokyta and Ruzicka [MNRR96], and Fuchs and Seregin [FS00].

In the whole paper, we will concentrate on the following types of shear thickening fluids. To be precise, the potential h satisfies the following conditions:

$$\begin{aligned} &h \text{ is strictly increasing and convex} \\ &\text{together with } h''(0) > 0 \text{ and } \lim_{t \rightarrow 0} \frac{h(t)}{t} = 0. \end{aligned} \tag{A1}$$

$$\begin{aligned} &(\text{doubling property}) \text{ there exists a constant } a \\ &\text{such that } h(2t) \leq ah(t) \text{ for all } t \geq 0. \end{aligned} \tag{A2}$$

$$\text{we have } \frac{h'(t)}{t} \leq h''(t) \text{ for any } t \geq 0. \tag{A3}$$

The study of Liouville type of theorems for Navier-Stokes equations goes back to the work of Gilbarg and Weinberger [GW78]. They showed, among the others, that the entire solutions u of stationary Navier-Stokes equations in the plane are constants under the condition: $\int_{\mathbb{R}^2} |Du|^2 dx < \infty$. For the unstationary backward Navier-Stokes equations in 2D, recently, Koch, Nadirashvili, Seregin and Sverak [KNSS09] showed that $u(x, t) = b(t)$ on $\mathbb{R}^2 \times (-\infty, 0)$ provided the solutions are bounded. Clearly, this result implies the Liouville theorem for stationary Navier-Stokes equations, that is, bounded solutions to stationary Navier-Stokes equations in 2D are constants.

For general potential h satisfying (A1) – (A3), very recently Fuchs [Fuc] showed bounded solution u of (1.1) must be a constant vector provided the solution satisfies the asymptotic behavior $|u - u_\infty| \rightarrow 0$ at infinity, where u_∞ is a constant vector. Later, the author removed the above assumption on u at infinity and showed that every bounded solution u of (1.1) must be a constant vector in [Zha].

Under different hypothesis that the flow is slow (which means the convective term vanishes) and energy is finite, i.e. $\int_{\mathbb{R}^2} h(|Du|) dx < \infty$, Fuchs [Fuc] showed that the velocity field u is a constant vector. In this note, we remove the assumption that the flow is slow and prove the following theorem which is the analogue result of Gilbarg and Weinberger in the setting of shear thickening fluids.

Theorem 1.1. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution to (1.1) i.e.*

$$(1.4) \quad \int_{\mathbb{R}^2} T(\varepsilon(u)) : \varepsilon(\varphi) dx - \int_{\mathbb{R}^2} u^k u^i \partial_k \varphi^i dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$, $\operatorname{div} \varphi = 0$, and satisfy the condition $\int_{\mathbb{R}^2} h(|Du|) dx < \infty$. Then u is a constant vector.

Next, we consider another type of Liouville theorems for the solutions of (1.1). Recently, Fuchs [Fuc] showed that the solution is identically zero under the conditions: $\int_{\mathbb{R}^2} h(|\varepsilon(u)|)dx < \infty$ and $\int_{\mathbb{R}^2} |u|^2 dx < \infty$. Now we improve this result and obtain the following types of Liouville theorems.

Before stating the results, let us introduce some notations. It follows from (A1) – (A3) that there is a number $\tau \in (1, 2]$ such that

$$h'(t) \leq C(h(t)^{\frac{1}{\tau}} + 1),$$

where $C > 0$ is a constant, see Lemma 2.1 in section 2. We denote τ' by its Hölder conjugate exponent, $\tau' = \frac{\tau}{\tau-1}$. Clearly, for the Navier-Stokes model, i.e., $h(t) = \frac{\nu}{2}t^2$, $\tau = 2$ and $\tau' = 2$, and for the Ladyzhenskaya model, i.e., $h(t) = \frac{\nu}{2}t^2 + \mu t^p$, where $p > 2$, $\tau = \frac{p}{p-1}$ and $\tau' = p$.

Theorem 1.2. *Suppose that the potential h satisfies the conditions (A1) – (A3). Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution to equation (1.1).*

- (i) *Suppose that $3/2 \leq \tau \leq 2$. Let p be a number such that $p > 1$. If $u \in L^p(\mathbb{R}^2, \mathbb{R}^2)$, then u is the zero vector.*
- (ii) *Suppose that $4/3 < \tau < 3/2$. Let p be a number such that $p > \tau'$. If $u \in L^p(\mathbb{R}^2, \mathbb{R}^2)$, then u is the zero vector.*
- (iii) *Suppose that $1 < \tau \leq 4/3$. If $u \in L^{\tau'}(\mathbb{R}^2, \mathbb{R}^2)$, then u is the zero vector.*

In the setting of stationary Navier-Stokes equations, we have the following corollary.

Corollary 1.3. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution to stationary Navier-Stokes equations in the plane, i.e.*

$$\int_{\mathbb{R}^2} Du : D\varphi dx - \int_{\mathbb{R}^2} u^k u^i \partial_k \varphi^i dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$, $\operatorname{div} \varphi = 0$, and $u \in L^p(\mathbb{R}^2, \mathbb{R}^2)$, $p > 1$. Then u must be a zero vector.

We first comment on the regularity assumption on the solutions. As we known, for general h satisfying (A1) – (A3), $C^{1,\alpha}$ regularity of solutions is an open problem. But in some special case, such as $h(t) = t^2(1+t)^m$, $m \geq 0$, the solution u belongs to the space $C^{1,\alpha}$ [BFZ05]. For the further discussion about regularity of the solutions of (1.1) the readers are referred to see the paper [Fuc12]. Therefore the regularity assumptions on u in the above theorems are reasonable.

Second, we comment about the proofs of the theorems. For obtaining Theorem 1.1, in the special case $h(t) = \frac{\nu}{2}t^2$, Gilbarg and Weinberger's [GW78] approach relies on the following fact: the vorticity $\omega = \partial_{x_1} u^2 - \partial_{x_2} u^1$ satisfies the elliptic equation $-\Delta \omega + u \cdot D\omega = 0$ and hence it satisfies the maximum principle. In our setting for equation (1.1), it seems that this approach does not work. We follow the approach of Fuchs in [Fuc], see also [FZ12] and [Zha]. The essential idea is to study the energy estimate for the second order

derivatives, see Lemma 3.1 in Section 3. Then we conclude that u must be a constant vector under the condition $\int_{\mathbb{R}^2} h(|Du|)dx < \infty$.

The idea for proving Theorem 1.2 is as follows: if p is “suitable” small, we directly use the local energy estimate for the first order derivatives to control the local integral of $|u|^q$, where q is large enough, and conclude that $\int_{\mathbb{R}^2} |u|^q dx = 0$. If p is “suitable” large, we prove the local uniformly finite energy estimate for the second order derivatives, from which follows the boundedness of solutions. Then we conclude the proof of Theorem 1.2 by the Liouville theorem for bounded solutions in [Zha].

Our notations is standard. Throughout this paper, the convention of summation with respect to indices repeated twice is used. All constants are denoted by the symbol C , and C may change from line to line, whenever it is necessary we will indicate the dependence of C on parameters. As usual $Q_R(x_0)$ denotes the open square with center x_0 and side length $2R$, and symbols $:$, \cdot will be used for the scalar product of matrices and vectors respectively. $|\cdot|$ denotes the associated Eculidean norms.

Our paper is organized as follows: In section 2, we present some auxiliary results. In section 3, we give the proof of Theorem 1.1, and in section 4, we give the proof of Theorem 1.2.

2 Auxiliary Results

2.1 The properties of function h

The following properties of function h follow from (A1) – (A3), see [Fuc].

- (i) $\mu(t) = \frac{h'(t)}{t}$ is an increasing function.
- (ii) We have $h(0) = h'(0) = 0$ and

$$(2.1) \quad h(t) \geq \frac{1}{2}h''(0)t^2.$$

Moreover,

$$(2.2) \quad \frac{h'(t)}{t} \geq \lim_{s \rightarrow 0} \frac{h'(s)}{s} = h''(0) > 0.$$

- (iii) It satisfies the balancing condition, i.e., for some $a > 0$,

$$(2.3) \quad \frac{1}{a}h'(t)t \leq h(t) \leq th'(t), \quad t \geq 0.$$

- (iv) For an exponent $m \geq 2$ and a constant $C \geq 0$ it holds

$$(2.4) \quad h(t) \leq C(1 + t^m), \quad h'(t) \leq C(1 + t^m), \quad t \geq 0.$$

From the assumptions on h , we know the system satisfies the following elliptic condition, $\forall \varepsilon, \sigma \in S^2$,

$$(2.5) \quad \frac{h'(|\varepsilon|)}{|\varepsilon|} |\sigma|^2 \leq D^2 H(\varepsilon)(\sigma, \sigma) \leq h''(|\varepsilon|) |\sigma|^2,$$

from which, together with (2.2), it follows that

$$(2.6) \quad D^2 H(\varepsilon)(\sigma, \sigma) \geq h''(0) |\sigma|^2.$$

Finally, we have the following lemma which is taken from [Fuc].

Lemma 2.1. *There is a number $\tau \in (1, 2]$ such that*

$$h'(t) \leq C(h(t)^{\frac{1}{\tau}} + 1)$$

or equivalently

$$|DH(\varepsilon(u))| \leq C(H(\varepsilon(u))^{\frac{1}{\tau}} + 1)$$

holds for all $t \geq 0$ and $\varepsilon \in S^2$. Moreover, we have the sharper estimate

$$h'(t) \leq C(h(t)^{\frac{1}{\tau}} + t), \quad t \geq 0.$$

2.2 Divergence equations and Korn's inequality

First, we introduce a standard result concerning the “divergence equations”, see e.g. [Gal94a], [Gal94b] or [FS00]. For any $R > 0$ and $x_0 \in \mathbb{R}^2$, define $Q_R(z) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid |\tilde{x} - x| < R, |\tilde{y} - y| < R, z = (x, y)\}$.

Lemma 2.2. *Consider a function $f \in L^q(Q_R(z))$, $q > 1$ such that $\int_{Q_R(z)} f dx = 0$. Then there exists a field $v \in W_0^{1,q}(Q_R(z), \mathbb{R}^2)$ and a constant $C(q)$, independent of $Q_R(z)$, such that we have $\operatorname{div} v = f$ on $Q_R(z)$ together with the estimate*

$$\int_{Q_R(z)} |Dv|^q dx \leq C(q) \int_{Q_R(z)} |f|^q dx.$$

Second, the following lemma is the classical Korn inequality, see [Tem83].

Lemma 2.3. *There is an absolute constant C such that for all $v \in W_0^{1,2}(Q_R(z), \mathbb{R}^2)$ it holds*

$$\int_{Q_R(z)} |Dv|^2 dx \leq C \int_{Q_R(z)} |\varepsilon(v)|^2 dx,$$

and for all $v \in W^{1,2}(Q_R(z), \mathbb{R}^2)$ it holds

$$\int_{Q_R(z)} |Dv|^2 dx \leq C \left(\int_{Q_R(z)} |\varepsilon(v)|^2 dx + \frac{1}{R^2} \int_{Q_R(z)} |v|^2 dx \right).$$

2.3 Ladyzhenskaya's inequality and Sobolev-Poincaré's inequality

First, we introduce a local version of the Sobolev inequality in the plane, see [GT83].

Lemma 2.4. *Let $x_0 \in \mathbb{R}^2$, $R > 0$, $Q_R(x_0) \subset \mathbb{R}^2$ and $u \in W^{1,2}(Q_R(x_0))$. Then, $\forall q > 1$, there exists a constant $C(q)$ depending only on q such that the following inequality holds*

$$\left(\frac{1}{R^2} \int_{Q_R(x_0)} |u|^q dx \right)^{\frac{1}{q}} \leq C(q) \left\{ \left(\int_{Q_R(x_0)} |Du|^2 dx \right)^{\frac{1}{2}} + \left(\frac{1}{R^2} \int_{Q_R(x_0)} |u|^2 dx \right)^{\frac{1}{2}} \right\}.$$

Next, we need a local version of Ladyzhenskaya's inequality. It is an easy consequence of Ladyzhenskaya's inequality. We give the proof here.

Lemma 2.5. *Suppose $u \in W^{1,2}(Q_R(x_0))$, $Q_R(x_0) \subset \mathbb{R}^2$. Then there exists a constant C_0 independent of R , x_0 such that*

$$\int_{Q_R(x_0)} |u|^4 dx \leq C_0 \left\{ \int_{Q_R(x_0)} |u|^2 dx \int_{Q_R(x_0)} |Du|^2 dx + \frac{1}{R^2} \left(\int_{Q_R(x_0)} |u|^2 dx \right)^2 \right\}.$$

Proof. For any $x_0 \in \mathbb{R}^2$ and $v \in W^{1,2}(Q_1(x_0))$, then, there exists an extension $\tilde{v} \in W_0^{1,2}(Q_2(x_0))$ of v s.t.

$$(2.7) \quad \|\tilde{v}\|_{L^2(Q_2(x_0))} \leq C \|v\|_{L^2(Q_1(x_0))}$$

and

$$(2.8) \quad \|\tilde{v}\|_{W_0^{1,2}(Q_2(x_0))} \leq C \|v\|_{W^{1,2}(Q_1(x_0))},$$

where C is an absolute constant. See [Eva98].

Moreover, by Ladyzhenskaya's inequality (see [Tem84]) we have

$$(2.9) \quad \int_{Q_2(x_0)} |\tilde{v}|^4 dx \leq 2 \int_{Q_2(x_0)} |\tilde{v}|^2 dx \int_{Q_2(x_0)} |D\tilde{v}|^2 dx.$$

Combing the estimates (2.7), (2.8) and (2.9), we obtain that

$$\begin{aligned} \int_{Q_2(x_0)} |\tilde{v}|^4 dx &\leq C \left(\int_{Q_1(x_0)} |Dv|^2 dx + \int_{Q_1(x_0)} |v|^2 dx \right) \int_{Q_1(x_0)} |v|^2 dx \\ &\leq C \left\{ \int_{Q_1(x_0)} |v|^2 dx \int_{Q_1(x_0)} |Dv|^2 dx + \left(\int_{Q_1(x_0)} |v|^2 dx \right)^2 \right\}, \end{aligned}$$

from which, it follows that

$$(2.10) \quad \int_{Q_1(x_0)} |v|^4 dx \leq C \left\{ \int_{Q_1(x_0)} |v|^2 dx \int_{Q_1(x_0)} |Dv|^2 dx + \left(\int_{Q_1(x_0)} |v|^2 dx \right)^2 \right\}.$$

For $R > 0$, let $v(x) := u(Rx)$, thus, $v(x) \in W^{1,2}(Q_1(x_0))$. Thus (2.10) holds, and hence, we end up with

$$\int_{Q_R(x_0)} |u|^4 dx \leq C \left\{ \int_{Q_R(x_0)} |u|^2 dx \int_{Q_R(x_0)} |Du|^2 dx + \frac{1}{R^2} \left(\int_{Q_R(x_0)} |u|^2 dx \right)^2 \right\}.$$

This finishes the proof. \square

2.4 A lemma of Gilbarg and Weinberger

The following result was due to Gilbarg and Weinberger, see Lemma 2.1 in [GW78].

Lemma 2.6. *Let $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ in $r > r_0 > 0$ and have finite integral*

$$\int_{r>r_0} |Df|^2 dx dy < \infty.$$

Then

$$\lim_{r \rightarrow \infty} \frac{1}{\log r} \int_0^{2\pi} f(r, \theta)^2 d\theta = 0.$$

2.5 A lemma of Giaquinta and Modica

The following ε -lemma goes back to the work of Giaquinta and Modica [GM82]. Recently, a generalized version of ε -lemma was given in [FZ12]. For proving our results, we need the following version of ε -lemma.

Lemma 2.7. *Let f, f_1, \dots, f_l denote non-negative functions from the space $L^1_{loc}(\mathbb{R}^2)$. Suppose further that we are given exponents $\alpha_1, \dots, \alpha_l \geq 0$, $\beta_1, \dots, \beta_l \geq 1$. For any $x_0 \in \mathbb{R}^2$, $Q = Q_{2R}(x_0)$, we can find δ_0 depending on $\alpha_1, \dots, \alpha_l \geq 0$ as follows: if for $\delta \in (0, \delta_0)$ it is possible to calculate a constant $C(\delta) > 0$ such that the inequality*

$$(2.11) \quad \int_{Q_{2r}(z)} f dx \leq \delta \int_{Q_{2r}(z)} f dx + C(\delta) \sum_{j=1}^l r^{-\alpha_j} \left(\int_{Q_{2r}(z)} f_j dx \right)^{\beta_j}$$

holds for any choice $Q_{2r}(z) \subset Q_{2R}(x_0)$. Then there is a constant C independent of δ and R with the property

$$\int_{Q_R(x_0)} f dx \leq C \sum_{j=1}^l R^{-\alpha_j} \left(\int_{Q_{2R}(x_0)} f_j dx \right)^{\beta_j}.$$

Remark 2.8. When $\beta_j = 1, j = 1, 2, \dots, l$, Lemma 2.7 is reduced to Lemma 3.1 of [FZ12]. Notice we have the trivial inequality $(\int_{Q_{2r}(z)} f_j dx)^{\beta_j} \leq (\int_{Q_{2R}(x_0)} f_j dx)^{\beta_j-1} \int_{Q_{2r}(z)} f_j dx$. In this way, we can reduce the assumption (2.11) to that of Lemma 3.1 of [FZ12]. Then the proof of Lemma 2.7 is exactly the same as that of Lemma 3.1 of [FZ12].

2.6 A Liouville theorem

We need the following Liouville theorem, Theorem 1 in [Zha], for the equation (1.1), to prove Theorem 1.2.

Theorem 2.9. *Suppose $u \in C^1(\mathbb{R}^2, \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution of (1.1). Then u is a constant vector.*

3 Proof of Theorem1.1

In view of $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and the elliptic condition(2.5), by standard difference quotient technique we can prove that $u \in W_{loc}^{2,2}(\mathbb{R}^2, \mathbb{R}^2)$. See [Fuc], [Zha]. The Proof of Theorem1.1 is divided into the following three lemmas.

Lemma 3.1. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution of (1.1) and satisfy the condition $\int_{\mathbb{R}^2} h(|Du|)dx < \infty$. Then, for any $x_0 \in \mathbb{R}^2$, $R > 0$, the following energy estimate holds*

$$(3.1) \quad \int_{Q_R(x_0)} W dx \leq C \left\{ \frac{1}{R^2} \int_{Q_{2R}(x_0)} h(|\varepsilon(u)|) dx + \frac{1}{R^2} \int_{Q_{2R}(x_0)} |Du|^2 dx + \left(1 + \frac{1}{R^{2m}}\right) + \frac{1}{R^3} \int_{Q_{2R}(x_0)} |u| dx \right\},$$

where $W = D^2 H(\varepsilon(u))(\varepsilon(\partial_k u), \varepsilon(\partial_k u))$, $m > 0$, C is a constant independent of x_0 , R .

Proof. For any cut-off function $\eta \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \eta \leq 1$, the following estimate is obtained in [Zha], see (3.9) of [Zha],

$$(3.2) \quad \begin{aligned} & \int_{\mathbb{R}^2} D^2 H(\varepsilon(u))(\varepsilon(\partial_k u), \varepsilon(\partial_k u)) \eta^2 dx \\ & \leq C \left\{ \int_{\mathbb{R}^2} h(|\varepsilon(u)|) |D\eta|^2 dx + \int_{\mathbb{R}^2} h'(|\varepsilon(u)|)^2 (|D\eta|^2 + |D^2 \eta|) dx \right. \\ & \quad \left. + \int_{\mathbb{R}^2} |Du|^2 (|D\eta|^2 + |D^2 \eta|) dx + \int_{\mathbb{R}^2} |Du|^2 |u| |D\eta| dx \right\}. \end{aligned}$$

Now, for any $x \in Q_{2R}(x_0)$, $r > 0$, $Q_{2r}(x) \subset Q_{2R}(x_0)$ and $\eta \in C_0^\infty(Q_{\frac{3}{2}r}(x))$ satisfying $\eta = 1$ in $Q_r(x)$ and $0 \leq \eta \leq 1$, $|D\eta| \leq \frac{4}{r}$, $|D^2 \eta| \leq \frac{16}{r^2}$, we deduce from (3.2) that

$$(3.3) \quad \begin{aligned} \int_{Q_r(x)} W dx & \leq C \left\{ \frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} h(|\varepsilon(u)|) dx + \frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} h'(|\varepsilon(u)|)^2 dx \right. \\ & \quad \left. + \frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} |Du|^2 dx + \frac{1}{r} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 |u| dx \right\}, \end{aligned}$$

where $T_{\frac{3}{2}r}(x) = Q_{\frac{3}{2}r}(x) \setminus \overline{Q_r(x)}$.

For the term $\frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} h'(|\varepsilon(u)|)^2 dx$, we have the following estimate, for $L > 0$, see (3.16) in [Zha],

$$\begin{aligned} \frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} h'(|\varepsilon(u)|)^2 dx & \leq C h'(L)^2 + C \frac{1}{L^2} \frac{1}{r^4} \left(\int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx \right)^2 \\ & \quad + C \frac{1}{r^2} \frac{1}{L^2} \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx \int_{Q_{2r}(x)} W dx. \end{aligned}$$

Choosing $L = \frac{1}{\varepsilon^{\frac{1}{2}}r}$, $\varepsilon < 1$, we have that

$$(3.4) \quad \begin{aligned} \frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} h'(|\varepsilon(u)|)^2 dx &\leq Ch'(\frac{1}{\varepsilon^{\frac{1}{2}}r})^2 + C \frac{1}{r^2} \left(\int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx \right)^2 \\ &+ C\varepsilon \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx \int_{Q_{2r}(x)} W dx. \end{aligned}$$

We then deal with the last term in (3.3). Letting $A = \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx$ and $B = \int_{T_{\frac{3}{2}r}(x)} u dx$, we have

$$\begin{aligned} \frac{1}{r} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 |u| dx &\leq \frac{1}{r} \int_{T_{\frac{3}{2}r}(x)} ||Du|^2 - A| |u - B| dx \\ &+ \frac{1}{r} |B| \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx + \frac{1}{r} |A| \int_{T_{\frac{3}{2}r}(x)} |u - B| dx. \end{aligned}$$

Recalling the choices of A and B , we obtain by Young's inequality, for $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{r} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 |u| dx &\leq \varepsilon \int_{T_{\frac{3}{2}r}(x)} ||Du|^2 - A|^2 dx + \frac{1}{\varepsilon} \frac{1}{r^2} \int_{T_{\frac{3}{2}r}(x)} |u - B|^2 dx \\ &+ \frac{C}{r^3} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx \int_{T_{\frac{3}{2}r}(x)} |u| dx. \end{aligned}$$

Applying Poincaré's inequality and Sobolev-Poincaré's inequality we obtain that

$$\begin{aligned} \frac{1}{r} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 |u| dx &\leq \varepsilon \left(\int_{T_{\frac{3}{2}r}(x)} |D(|Du|^2)| dx \right)^2 + \frac{1}{\varepsilon} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx \\ &+ \frac{C}{r^3} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx \int_{T_{\frac{3}{2}r}(x)} |u| dx, \end{aligned}$$

from which, together with Hölder's inequality, it follows that

$$(3.5) \quad \begin{aligned} \frac{1}{r} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 |u| dx &\leq \varepsilon \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx \int_{T_{\frac{3}{2}r}(x)} |D^2 u|^2 dx \\ &+ \frac{1}{\varepsilon} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx + \frac{C}{r^3} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx \int_{T_{\frac{3}{2}r}(x)} |u| dx. \end{aligned}$$

Combining (3.3), (3.4) and (3.5) and observing the inequality $|D^2 u(x)| \leq C |D\varepsilon(u)(x)| \leq$

$CW(x)$ we deduce that

$$\begin{aligned}
\int_{Q_r(x)} W dx &\leq C\varepsilon \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx \int_{Q_{2r}(x)} W dx \\
&\quad + C\varepsilon \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx \int_{Q_{2r}(x)} W dx \\
(3.6) \quad &\quad + \frac{C}{r^2} \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx + \frac{C}{r^2} \int_{Q_{2r}(x)} |Du|^2 dx \\
&\quad + Ch'(\frac{1}{\varepsilon^{\frac{1}{2}r}})^2 + \frac{C}{r^2} (\int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx)^2 \\
&\quad + \frac{C}{\varepsilon} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx + \frac{C}{r^3} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx \int_{T_{\frac{3}{2}r}(x)} |u| dx.
\end{aligned}$$

Since $\int_{\mathbb{R}^2} |Du|^2 dx \leq C \int_{\mathbb{R}^2} h(|Du|) dx < \infty$, choosing ε small enough and denoting $\delta := C\varepsilon < \frac{1}{2}$, we obtain from (3.6) that

$$\begin{aligned}
\int_{Q_r(x)} W dx &\leq \delta \int_{Q_{2r}(x)} W dx + \frac{C}{r^2} \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx + \frac{C}{r^2} \int_{Q_{2r}(x)} |Du|^2 dx \\
(3.7) \quad &\quad + C \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx + Ch'(\frac{1}{\varepsilon^{\frac{1}{2}r}})^2 + \frac{C}{r^3} \int_{Q_{\frac{3}{2}r}(x)} |u| dx.
\end{aligned}$$

In view of the condition $h'(t) \leq C(1+t^m)$ and $\int_{\mathbb{R}^2} |Du|^2 dx \leq C \int_{\mathbb{R}^2} h(|Du|) dx < \infty$ it follows that

$$\begin{aligned}
\int_{Q_r(x)} W dx &\leq \delta \int_{Q_{2r}(x)} W dx + \frac{C}{r^2} \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx + \frac{C}{r^2} \int_{Q_{2r}(x)} |Du|^2 dx \\
&\quad + C(1 + \frac{1}{r^{2m}}) + \frac{C}{r^3} \int_{Q_{2r}(x)} |u| dx.
\end{aligned}$$

By Lemma 2.7 we end up with

$$\begin{aligned}
\int_{Q_R(x_0)} W dx &\leq C \left\{ \frac{1}{R^2} \int_{Q_{2R}(x_0)} h(|\varepsilon(u)|) dx + \frac{1}{R^2} \int_{Q_{2R}(x_0)} |Du|^2 dx \right. \\
&\quad \left. + \left(1 + \frac{1}{R^{2m}}\right) + \frac{1}{R^3} \int_{Q_{2R}(x_0)} |u| dx \right\}.
\end{aligned}$$

□

Lemma 3.2. *Let u be as in Lemma 3.1. Then the following estimate holds*

$$(3.8) \quad \int_{\mathbb{R}^2} D^2 H(\varepsilon(u))(\varepsilon(\partial_k u), \varepsilon(\partial_k u)) dx < \infty.$$

Therefore,

$$(3.9) \quad \int_{\mathbb{R}^2} |D^2 u|^2 dx < \infty.$$

Proof. Since $\int_{\mathbb{R}^2} h(|Du|)dx =: M < \infty$, for $R > 1$, (3.1) gives

$$(3.10) \quad \int_{Q_R(x_0)} W dx \leq C(M) + \frac{C}{R^3} \int_{Q_{2R}(x_0)} |u| dx.$$

Since $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $\int_{\mathbb{R}^2} |Du|^2 dx \leq C \int_{\mathbb{R}^2} h(|Du|) dx < \infty$, using Lemma 2.6 we deduce that

$$(3.11) \quad \limsup_{R \rightarrow \infty} \frac{1}{R^3} \int_{Q_{2R}(x_0)} |u| dx = 0.$$

Letting $R \rightarrow \infty$ in (3.10) we have

$$(3.12) \quad \int_{\mathbb{R}^2} W dx < \infty.$$

Since $|D^2u(x)| \leq C|D\varepsilon(u)(x)|$, then (3.12) implies (3.9). The proof is complete. \square

Lemma 3.3. *Let u be as in Lemma 3.1, then we have*

$$(3.13) \quad \int_{\mathbb{R}^2} W dx = 0.$$

Hence, u must be a constant vector.

Proof. Since $\int_{\mathbb{R}^2} |Du|^2 dx \leq C \int_{\mathbb{R}^2} h(|Du|) dx < \infty$, it implies that $\lim_{r \rightarrow \infty} \int_{T_{\frac{3}{2}r}(x)} |Du|^2 dx = 0$. Letting $r \rightarrow \infty$ in (3.7), we have by the condition $h'(0) = 0$, (3.11) and (3.12) that

$$(3.14) \quad \int_{\mathbb{R}^2} W dx \leq \frac{1}{2} \int_{\mathbb{R}^2} W dx.$$

Thus (3.13) holds, and hence $W(x) = 0$. From the relation $|D^2u(x)|^2 \leq C|D\varepsilon(u)(x)|^2 \leq CW(x)$, we know that $D^2u(x) = 0$. Therefore u must be an affine function. On the other hand, in view of inequality $\int_{\mathbb{R}^2} |Du|^2 dx \leq C \int_{\mathbb{R}^2} h(|Du|) dx < \infty$, it gives $Du(x) = 0$. Then, u is a constant vector. \square

4 Proof of 1.2

To prove Theorem 1.2, the Liouville theorem under the integrability condition of u , we need the energy estimates for the first order derivatives and the second order derivatives. The following Lemma gives that for the first order derivatives.

Lemma 4.1. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution of (1.1). Then, for any $x_0 \in \mathbb{R}^2$, $R > 0$, the following energy estimate holds*

$$(4.1) \quad \int_{Q_R(x_0)} h(|\varepsilon(u)|) dx \leq C \left\{ \frac{1}{R^{\tau'}} \int_{Q_{2R}(x_0)} |u|^{\tau'} dx + \frac{1}{R^2} \int_{Q_{2R}(x_0)} |u|^2 dx + \frac{1}{R^2} \left(\int_{Q_{2R}(x_0)} |u|^2 dx \right)^2 \right\},$$

where $\tau' = \frac{\tau}{\tau-1}$ and $\tau, 1 < \tau \leq 2$, is as in Lemma 2.1.

Proof. For any $x_0 \in \mathbb{R}^2$, $R > 0$ and $x \in Q_{2R}(x_0)$, $r > 0$ s.t. $Q_{2r}(x) \subset Q_{2R}(x_0)$, we choose the cut-off function $\eta \in C_0^\infty(Q_{\frac{3}{2}r}(x))$ as Lemma 3.1 and find a solution ϖ to the following equation

$$\operatorname{div} \varpi = \operatorname{div}(u\eta^2) = u \cdot D\eta^2$$

s.t.

$$\operatorname{spt} \varpi \subset Q_{\frac{3}{2}r}(x)$$

and for any $q > 1$, the following estimate holds

$$(4.2) \quad \|\varpi\|_{W^{1,q}} \leq C(q) \|u \cdot D\eta^2\|_{L^q}.$$

Taking the test function $\varphi = u\eta^2 - \varpi$ in (1.1) we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} DH(\varepsilon(u)) : \varepsilon(u\eta^2) dx - \int_{\mathbb{R}^2} DH(\varepsilon(u)) : \varepsilon(\varpi) dx - \int_{\mathbb{R}^2} u^i u^j \\ \partial_i(u^j \eta^2) dx + \int_{\mathbb{R}^2} u^i u^j \partial_i \varpi^j dx = 0. \end{aligned}$$

Hence,

$$(4.3) \quad \begin{aligned} \int_{\mathbb{R}^2} DH(\varepsilon(u)) : \varepsilon(u)\eta^2 dx &= - \int_{\mathbb{R}^2} DH(\varepsilon(u)) : u \otimes D\eta^2 dx + \int_{\mathbb{R}^2} DH(\varepsilon(u)) \\ &: \varepsilon(\varpi) dx + \int_{\mathbb{R}^2} u^i u^j \partial_i(u^j \eta^2) dx - \int_{\mathbb{R}^2} u^i u^j \partial_i \varpi^j dx \\ &=: I + II + III + IV. \end{aligned}$$

Recalling the definition of H and η and applying Lemma 2.1 and Young's inequality, we have for any $0 < \delta < 1$

$$(4.4) \quad \begin{aligned} I &\leq \int_{\mathbb{R}^2} h'(|\varepsilon(u)|) |u| |D\eta| dx \leq C \int_{\mathbb{R}^2} \left(h(|\varepsilon(u)|)^{\frac{1}{\tau}} + |\varepsilon(u)| \right) |u| |D\eta| dx \\ &\leq \delta \int_{Q_{\frac{3}{2}r}(x)} h(|\varepsilon(u)|) dx + C(\tau, \delta) \left\{ \frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx + \frac{1}{r^{\tau'}} \int_{Q_{\frac{3}{2}r}(x)} |u|^{\tau'} dx \right\}. \end{aligned}$$

We deal with II in the same way and we obtain

$$\begin{aligned} II &\leq \delta \int_{Q_{\frac{3}{2}r}(x)} h(|\varepsilon(u)|) dx + C(\tau, \delta) \left\{ \int_{Q_{\frac{3}{2}r}(x)} |\varepsilon(\varpi)|^2 dx \right. \\ &\quad \left. + \int_{Q_{\frac{3}{2}r}(x)} |\varepsilon(\varpi)|^{\tau'} dx \right\}. \end{aligned}$$

Thus it follows from (4.2) that

$$(4.5) \quad II \leq \delta \int_{Q_{\frac{3}{2}r}(x)} h(|\varepsilon(u)|) dx + C(\tau, \delta) \left\{ \frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx + \frac{1}{r^{\tau'}} \int_{Q_{\frac{3}{2}r}(x)} |u|^{\tau'} dx \right\}.$$

We estimate III by integration by parts. We have by the equation $\operatorname{div} u = 0$ and Young's inequality that for any $\varepsilon > 0$,

$$\begin{aligned} III &= \int_{\mathbb{R}^2} u^i u^j \partial_i (u^j \eta^2) dx = \int_{\mathbb{R}^2} u^i u^j \partial_i u^j \eta^2 dx + \int_{\mathbb{R}^2} u^i u^j u^j \partial_i \eta^2 dx \\ &= \int_{\mathbb{R}^2} \frac{|u|^2}{2} u \cdot D\eta^2 \leq \frac{C}{r} \int_{Q_{\frac{3}{2}r}(x)} |u|^3 dx \\ &\leq \varepsilon \int_{Q_{\frac{3}{2}r}(x)} |u|^4 dx + \frac{C}{\varepsilon r^2} \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx. \end{aligned}$$

It remains to deal with IV . By Young's inequality and (4.2), we have

$$IV \leq \varepsilon \int_{Q_{\frac{3}{2}r}(x)} |u|^4 dx + \frac{C}{\varepsilon r^2} \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx.$$

Furthermore, Lemma 2.5 gives us

$$\int_{Q_{\frac{3}{2}r}(x)} |u|^4 dx \leq C_0 \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx \int_{Q_{\frac{3}{2}r}(x)} |Du|^2 dx + \frac{C_0}{r^2} \left(\int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx \right)^2.$$

We Choose $\varepsilon = \frac{\delta}{C_0(1 + \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx)}$. Then III and IV are both bounded from

above by

$$(4.6) \quad \delta \int_{Q_{\frac{3}{2}r}(x)} |Du|^2 dx + C(\delta) \left\{ \frac{1}{r^2} \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx + \frac{1}{r^2} \left(\int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx \right)^2 \right\}.$$

Now, we want to give an appropriate control for the term $\int_{Q_{\frac{3}{2}r}(x)} |Du|^2 dx$. Choosing the cut-off function ξ s.t. $\xi \in C_0^\infty(Q_{2r}(x))$, $0 \leq \xi \leq 1$, $\xi = 1$ on $Q_{\frac{3}{2}r}(x)$ and $|D\xi| \leq \frac{4}{r}$, we deduce from Lemma 2.3 that

$$\begin{aligned} (4.7) \quad \int_{Q_{\frac{3}{2}r}(x)} |Du|^2 dx &\leq \int_{Q_{2r}(x)} |D(u\xi)|^2 dx \leq C \int_{Q_{2r}(x)} |\varepsilon(u\xi)|^2 dx \\ &\leq C \int_{Q_{2r}(x)} |\varepsilon(u)|^2 dx + \frac{C}{r^2} \int_{Q_{2r}(x)} |u|^2 dx. \end{aligned}$$

Putting together the estimates (4.3), (4.4), (4.5), (4.6) and (4.7) and noting $h(\varepsilon(u)) \geq \frac{1}{2} h''(0) |\varepsilon(u)|^2$, $h''(0) > 0$, we have

$$(4.8) \quad \begin{aligned} \int_{\mathbb{R}^2} DH(\varepsilon(u)) : \varepsilon(u) \eta^2 dx &\leq C\delta \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx + C(\delta, \tau) \left\{ \frac{1}{r^{\tau'}} \int_{Q_{\frac{3}{2}r}(x)} |u|^{\tau'} \right. \\ &\quad \left. dx + \frac{1}{r^2} \int_{Q_{2r}(x)} |u|^2 dx + \frac{1}{r^2} \left(\int_{Q_{2r}(x)} |u|^2 dx \right)^2 \right\}. \end{aligned}$$

Thus it follows from (2.3) that for any $\delta > 0$

$$\begin{aligned} \int_{Q_r(x)} h(|\varepsilon(u)|) dx &\leq C\delta \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx + C(\tau) \left\{ \frac{1}{r^{\tau'}} \int_{Q_{2r}(x)} |u|^{\tau'} dx \right. \\ &\quad \left. + \frac{1}{r^2} \int_{Q_{2r}(x)} |u|^2 dx + \frac{1}{r^2} \left(\int_{Q_{2r}(x)} |u|^2 dx \right)^2 \right\}, \end{aligned}$$

from which, by Lemma 2.7, it follows that

$$\begin{aligned} \int_{Q_R(x_0)} h(|\varepsilon(u)|) dx &\leq C \left\{ \frac{1}{R^{\tau'}} \int_{Q_{2R}(x_0)} |u|^{\tau'} dx + \frac{1}{R^2} \int_{Q_{2R}(x_0)} |u|^2 dx \right. \\ &\quad \left. + \frac{1}{R^2} \left(\int_{Q_{2R}(x_0)} |u|^2 dx \right)^2 \right\}. \end{aligned}$$

This finishes the proof of Lemma 4.1. \square

Lemma 4.2. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution of (1.1) and $2 \leq \tau' < 4$. Then, for any $x_0 \in \mathbb{R}^2$, $R > 0$ we have*

$$(4.9) \quad \begin{aligned} \int_{Q_R(x_0)} h(|\varepsilon(u)|) dx &\leq C(\tau) \left\{ \frac{1}{R^2} \int_{Q_{2R}(x_0)} |u|^2 dx + \frac{1}{R^{\bar{\tau}}} \int_{Q_{2R}(x_0)} |u|^2 dx \right. \\ &\quad \left. + \frac{1}{R^2} \left(\int_{Q_{2R}(x_0)} |u|^2 dx \right)^2 + \frac{1}{R^{\bar{\tau}}} \left(\int_{Q_{2R}(x_0)} |u|^2 dx \right)^{\tau^*+1} \right\}, \end{aligned}$$

where $\bar{\tau} = \frac{2\tau'}{4-\tau'} \geq 2$ and $\tau^* = \frac{\tau'-2}{4-\tau'}$.

Proof. Returning to (4.8), we just need to control $\frac{1}{r^{\tau'}} \int_{Q_{\frac{3}{2}r}(x)} |u|^{\tau'} dx$. By Young's inequality we have, for any $\varepsilon > 0$,

$$(4.10) \quad \frac{1}{r^{\tau'}} \int_{Q_{\frac{3}{2}r}(x)} |u|^{\tau'} dx \leq \varepsilon \int_{Q_{\frac{3}{2}r}(x)} |u|^4 dx + C(\tau) \frac{1}{\varepsilon^{\tau^*}} \frac{1}{r^{\bar{\tau}}} \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx,$$

where $\bar{\tau} = \frac{2\tau'}{4-\tau'}$. Moreover, by Lemma 2.5 we find

$$(4.11) \quad \int_{Q_{\frac{3}{2}r}(x)} |u|^4 dx \leq C_0 \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx \int_{Q_{\frac{3}{2}r}(x)} |Du|^2 dx + C_0 \frac{1}{r^2} \left(\int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx \right)^2.$$

Thus, putting together the estimates (4.10), (4.11) we have

$$(4.12) \quad \begin{aligned} \frac{1}{r^{\tau'}} \int_{Q_{\frac{3}{2}r}(x)} |u|^{\tau'} dx &\leq \varepsilon C_0 \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx \int_{Q_{\frac{3}{2}r}(x)} |Du|^2 dx \\ &\quad + \varepsilon C_0 \frac{1}{r^2} \left(\int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx \right)^2 + C(\tau) \frac{1}{\varepsilon^{\tau^*}} \frac{1}{r^{\bar{\tau}}} \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx. \end{aligned}$$

Letting $\varepsilon = \frac{\delta}{C_0(1 + \int_{Q_{\frac{3}{2}r}(x)} |u|^2 dx)}$ and putting together the estimates(4.8),(4.12), in view of $\eta = 1$ in $Q_r(x)$, we end up with

$$\begin{aligned} \int_{Q_r(x)} h(|\varepsilon(u)|) dx &\leq C\delta \int_{Q_{2r}(x)} h(|\varepsilon(u)|) dx + \delta \int_{Q_{\frac{3}{2}r}(x)} |Du|^2 dx \\ &\quad + C(\tau, \delta) \left\{ \frac{1}{r^2} \int_{Q_{2r}(x)} |u|^2 dx + \frac{1}{r^{\bar{\tau}}} \int_{Q_{2r}(x)} |u|^2 dx \right. \\ &\quad \left. + \frac{1}{r^2} \left(\int_{Q_{2r}(x)} |u|^2 dx \right)^2 + \frac{1}{r^{\bar{\tau}}} \left(\int_{Q_{2r}(x)} |u|^2 dx \right)^{\tau^*+1} \right\}. \end{aligned}$$

To prove (4.9), we may repeat the steps after(4.6) in the proof of Lemma4.1. We omit the details. \square

Lemma 4.3. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution of(1.1), then the following holds*

- (a) *If $2 \leq \tau' \leq 3$ and $u \in L^p(\mathbb{R}^2, \mathbb{R}^2)$, $1 < p < 2$, then u must be a zero vector.*
- (b) *If $3 < \tau' < 4$ and $u \in L^p(\mathbb{R}^2, \mathbb{R}^2)$, $\tau' - 2 < p < 2$, then u must be a zero vector.*

Proof. For any $q > 2$, by Lemma2.4 we have

$$\int_{Q_R(x_0)} |u|^q dx \leq C(q) \left\{ R^2 \left(\int_{Q_R(x_0)} |Du|^2 dx \right)^{\frac{q}{2}} + \frac{R^2}{R^q} \left(\int_{Q_R(x_0)} |u|^2 dx \right)^{\frac{q}{2}} \right\},$$

from which, together with Lemma2.3, it follows that

$$(4.13) \quad \int_{Q_R(x_0)} |u|^q dx \leq C(q) \left\{ R^2 \left(\int_{Q_R(x_0)} |\varepsilon(u)|^2 dx \right)^{\frac{q}{2}} + \frac{R^2}{R^q} \left(\int_{Q_R(x_0)} |u|^2 dx \right)^{\frac{q}{2}} \right\}.$$

Combining(4.9) and(4.13) we obtain

$$\begin{aligned} \int_{Q_R(x_0)} |u|^q dx &\leq C(q) \left\{ R^2 \left(\int_{Q_R(x_0)} h(\varepsilon(u)) dx \right)^{\frac{q}{2}} + \frac{R^2}{R^q} \left(\int_{Q_R(x_0)} |u|^2 dx \right)^{\frac{q}{2}} \right\} \\ (4.14) \quad &\leq C(\tau, q) \left\{ \frac{R^2}{R^q} \left(\int_{Q_{2R}(x_0)} |u|^2 dx \right)^{\frac{q}{2}} + \frac{R^2}{R^{\bar{\tau}\frac{q}{2}}} \left(\int_{Q_{2R}(x_0)} |u|^2 dx \right)^{\frac{q}{2}} \right. \\ &\quad \left. + \frac{R^2}{R^q} \left(\int_{Q_{2R}(x_0)} |u|^2 dx \right)^q + \frac{R^2}{R^{\bar{\tau}\frac{q}{2}}} \left(\int_{Q_{2R}(x_0)} |u|^2 dx \right)^{\frac{q}{2}(\tau^*+1)} \right\}. \end{aligned}$$

On the other hand, for $1 < p < 2$, Hölder's inequality gives

$$(4.15) \quad \int_{Q_{2R}(x_0)} |u|^2 dx \leq \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\frac{1}{p'}} \left(\int_{Q_{2R}(x_0)} |u|^q dx \right)^{\frac{1}{q'}},$$

where $p' = \frac{q-p}{q-2}$, $q' = \frac{q-p}{2-p}$.

Putting together the estimates(4.14) and(4.15) we deduce

$$\begin{aligned}
\int_{Q_R(x_0)} |u|^q dx &\leq C(\tau, q) \left\{ \frac{R^2}{R^q} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\frac{q}{2p'}} \left(\int_{Q_{2R}(x_0)} |u|^q dx \right)^{\frac{q}{2q'}} \right. \\
&\quad + \frac{R^2}{R^{\bar{\tau}\frac{q}{2}}} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\frac{q}{2p'}} \left(\int_{Q_{2R}(x_0)} |u|^q dx \right)^{\frac{q}{2q'}} \\
&\quad + \frac{R^2}{R^q} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\frac{q}{p'}} \left(\int_{Q_{2R}(x_0)} |u|^q dx \right)^{\frac{q}{q'}} \\
&\quad \left. + \frac{R^2}{R^{\bar{\tau}\frac{q}{2}}} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\frac{q(\tau^*+1)}{2p'}} \left(\int_{Q_{2R}(x_0)} |u|^q dx \right)^{\frac{q(\tau^*+1)}{2q'}} \right\}.
\end{aligned} \tag{4.16}$$

Notice that we always have $2 - p < 1$ and $\frac{2-p}{4-\tau'} < 1$, under our assumption (a) or (b) on τ' and p . Since $\lim_{q \rightarrow \infty} \frac{q}{q'} = 2 - p$ and $\lim_{q \rightarrow \infty} \frac{q(\tau^*+1)}{2q'} = \frac{2-p}{4-\tau'}$, we can choose q large enough s.t $\frac{q}{q'} < 1$, $\frac{q(\tau^*+1)}{2q'} < 1$. Thus, for any $\delta > 0$, using Young inequality in(4.16) we have

$$\begin{aligned}
\int_{Q_R(x_0)} |u|^q dx &\leq \delta \int_{Q_{2R}(x_0)} |u|^q dx + C(\tau, \delta, q) \left\{ \left(\frac{R^2}{R^q} \right)^{\alpha_1} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\beta_1} \right. \\
&\quad + \left(\frac{R^2}{R^{\bar{\tau}\frac{q}{2}}} \right)^{\alpha_2} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\beta_2} + \left(\frac{R^2}{R^q} \right)^{\alpha_3} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\beta_3} \\
&\quad \left. + \left(\frac{R^2}{R^{\bar{\tau}\frac{q}{2}}} \right)^{\alpha_4} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\beta_4} \right\},
\end{aligned}$$

where $\alpha_i, \beta_i, 1 \leq i \leq 4$, are positive numbers.

By Lemma2.7 we obtain

$$\begin{aligned}
\int_{Q_R(x_0)} |u|^q dx &\leq C(\tau, q) \left\{ \left(\frac{R^2}{R^q} \right)^{\alpha_1} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\beta_1} \right. \\
&\quad + \left(\frac{R^2}{R^{\bar{\tau}\frac{q}{2}}} \right)^{\alpha_2} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\beta_2} + \left(\frac{R^2}{R^q} \right)^{\alpha_3} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\beta_3} \\
&\quad \left. + \left(\frac{R^2}{R^{\bar{\tau}\frac{q}{2}}} \right)^{\alpha_4} \left(\int_{Q_{2R}(x_0)} |u|^p dx \right)^{\beta_4} \right\}.
\end{aligned} \tag{4.17}$$

Letting $R \rightarrow \infty$ in(4.17) and observing $\int_{\mathbb{R}^2} |u|^p dx < \infty$ we deduce that

$$\int_{\mathbb{R}^2} |u|^q dx = 0,$$

therefore, $u = 0$, and the proof is complete. \square

Lemma 4.4. *Let $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be an entire weak solution of(1.1), then the following results hold*

- (a) *If $2 \leq \tau' < 4$ and $u \in L^p(\mathbb{R}^2, \mathbb{R}^2)$, $p \geq 2$, then u must be a zero vector.*
- (b) *If $\tau' \geq 4$ and $u \in L^{\tau'}(\mathbb{R}^2, \mathbb{R}^2)$, then u must be a zero vector.*

Proof. For (a), we will first show that the integrals of $h(|\varepsilon(u)|)$ and $|Du|^2$ are both local uniformly bounded. In fact, for any $x_0 \in \mathbb{R}^2$, choosing $R = 2$ in (4.9) and recalling the condition (2.1) and (2.2) we obtain

$$\int_{Q_2(x_0)} |\varepsilon(u)|^2 dx \leq C \int_{Q_2(x_0)} h(|\varepsilon(u)|) dx \leq C(\|u\|_{L^p}, \tau), \forall x_0 \in \mathbb{R}^2,$$

from which, together with Lemma 2.3, it gives

$$\int_{Q_2(x_0)} |Du|^2 dx \leq C(\|u\|_{L^p}, \tau), \forall x_0 \in \mathbb{R}^2.$$

Now Lemma 3.1 gives us

$$\begin{aligned} \int_{Q_1(x_0)} W dx &\leq C(\|u\|_{L^p}, \tau) + C \int_{Q_2(x_0)} |u| dx \\ &\leq C(\|u\|_{L^p}, \tau). \end{aligned}$$

Thus by the equality $|D^2 u(x)| \leq C|D\varepsilon(u)(x)|$ we have

$$\int_{Q_1(x_0)} |D^2 u|^2 dx \leq C \int_{Q_1(x_0)} |D\varepsilon(u)|^2 dx \leq C \int_{Q_1(x_0)} W dx \leq C(\|u\|_{L^p}, \tau).$$

Now by Sobolev's imbedding theorem, we know $u \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Hence, by Lemma 2.9, u must be a constant vector. Since $u \in L^p(\mathbb{R}^2, \mathbb{R}^2)$, then $u = 0$.

The proof of (a) is complete. For (b), we prove in the same way. Since $u \in L^{\tau'}(\mathbb{R}^2, \mathbb{R}^2)$, Lemma 4.1 gives us the uniform estimate (4) in this case. The rest of the proof is exactly the same. This finishes the proof of Lemma 4.2. \square

Remark 4.5. If $u \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ is an entire weak solution of (1.1) and $\int_{\mathbb{R}^2} h(|\varepsilon(u)|) dx < \infty$, we can't deduce that u must be a constant vector. The counter example is $u_1 = -y, u_2 = x$. But if the integrals of $|u|^p, p \geq 1$ and $h(|\varepsilon(u)|)$ are both local uniformly bounded, then by Lemma 3.1 we know $u \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Hence u is a constant vector.

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